A COUNTEREXAMPLE TO MCKEE'S PENTANGULATION CONJECTURE

USING GPT 5.6 SOL PRO

Abstract. Let $G = C_7^2$ be the graph on $\mathbb{Z}_7$ in which i and j are adjacent exactly when $i - j \equiv \pm 1, \pm 2 \pmod 7$. We prove that every k-cycle of G, for $k \geq 5$, is the symmetric difference of $k-4$ distinct 5-cycles, whereas the Hamilton cycle 0123456 is not the symmetric difference of a 5-cycle and a 6-cycle. Thus G is pentangulated but not incrementally pentangulated, and McKee's conjecture is false.

1. Introduction

We identify a cycle with its edge set and write $\oplus$ for symmetric difference. McKee [1] called a graph *pentangulated* if every cycle C of length $k \geq 5$ is the binary sum of $k-4$ distinct 5-cycles. He called a graph *incrementally pentangulated* if every cycle C of length $k \geq 6$ has a representation

$$C = P \oplus Q,$$

where P is a 5-cycle and Q is a $(k-1)$-cycle. Every incrementally pentangulated graph is pentangulated, and McKee conjectured the converse; the problem is also recorded by West [2].

Theorem 1.1. *The graph C_7^2 is pentangulated but not incrementally pentangulated. Consequently, McKee's conjecture is false.*

The proof is elementary. Its only bookkeeping is the classification of the 6- and 7-cycles of a seven-vertex graph under a dihedral action.

2. The graph and its symmetries

Let G have vertex set $\mathbb{Z}_7$ and edge set

$$E(G) = \big\{\{i, j\} : i - j \equiv \pm 1, \pm 2 \pmod 7\big\}.$$

Thus $G = C_7^2$; multiplication by 2 on $\mathbb{Z}_7$ gives an isomorphism $G \cong \overline{C_7}$. A word such as 013542 denotes the cycle $0 - 1 - 3 - 5 - 4 - 2 - 0$.

Every map

$$x \longmapsto \varepsilon x + b \pmod 7, \qquad \varepsilon \in \{1, -1\}, \quad b \in \mathbb{Z}_7,$$

is an automorphism of G. These maps form a dihedral group D of order 14. Since G has seven vertices, proving pentangulation requires only the cycle lengths 5, 6, and 7.

3. Pentangulation

A 5-cycle is already a one-term binary sum. We treat the other two lengths separately.

3.1. **Six-cycles.** A 6-cycle omits one vertex, which by translation may be taken to be 6. The graph induced by $\{0, 1, 2, 3, 4, 5\}$ is K_6 with the five edges of the path

$$3 - 0 - 4 - 1 - 5 - 2$$

deleted. Inclusion-exclusion over these five forbidden edges gives

$$60 - 120 + 96 - 38 + 8 - 1 = 5$$

Hamilton cycles. The five cycles are

$$012345, \qquad 012435, \qquad 013245, \qquad 013542, \qquad 021345.$$

The reflection $x \mapsto 5 - x$ exchanges 012435 and 021345 and fixes the other three as edge sets. Translating the omitted vertex therefore yields the four D-orbits in Table 1. Every equality is an equality of edge sets modulo 2.

Table 1. The D-orbits of 6-cycles.

Orbit size	6-cycle	Binary sum of two distinct 5-cycles
7	012345	$01645 \oplus 12346$
14	012435	$01246 \oplus 05346$
7	013245	$01235 \oplus 12453$
7	013542	$01245 \oplus 02135$

All cycles on the right are 5-cycles of G, and the two in each row are distinct. The orbit sizes total 35, so every 6-cycle has the required decomposition.

3.2. **Seven-cycles.** The seven nonedges of G form a 7-cycle F. We count the Hamilton cycles of K_7 that avoid F. For $1 \le k \le 6$, if a k-edge subset of F has c path components, then the number of such subsets is

$$N_{k,c} = \frac{7}{c}\binom{k-1}{c-1}\binom{6-k}{c-1}.$$

After contracting its path components, the number of Hamilton cycles containing that subset is

$$2^c \frac{(6-k)!}{2}.$$

Together with the direct counts for $k = 0$ and $k = 7$, the aggregate inclusion-exclusion terms are

$$\begin{array}{c|cccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline T_k & 360 & 840 & 840 & 462 & 154 & 35 & 7 & 1. \end{array}$$

Hence G has

$$360 - 840 + 840 - 462 + 154 - 35 + 7 - 1 = 23$$

Hamilton cycles.

Those 23 cycles split into the five D-orbits in Table 2. The first and last representatives are fixed by all of D. The middle three are fixed, respectively, by the reflections $x \mapsto 4 - x$, $x \mapsto 1 - x$, and $x \mapsto 6 - x$, and by no nonidentity rotation; hence each of those orbits has size 7. The five orbits are distinct because their representatives use, respectively, $7, 5, 3, 2, 0$ edges of the rim 0123456.

Table 2. The D-orbits of 7-cycles.

Orbit size	7-cycle	Binary sum of three distinct 5-cycles
1	0123456	$01235 \oplus 01345 \oplus 01356$
7	0123465	$01235 \oplus 01346 \oplus 01356$
7	0132465	$01235 \oplus 01246 \oplus 01356$
7	0135642	$01235 \oplus 02135 \oplus 23564$
1	0246135	$01235 \oplus 01246 \oplus 02316$

Again, every cycle on the right is a 5-cycle of G, the three cycles in each row are pairwise distinct, and direct edge cancellation verifies each binary sum. The orbit sizes total 23, so the table covers every 7-cycle. Together with the 5- and 6-cycle cases, this proves that G is pentangulated.

4. Failure of incremental pentangulation

Let

$$R = 0123456.$$

Suppose, toward a contradiction, that $R = P \oplus Q$, where P is a 5-cycle and Q is a 6-cycle. Equivalently, $Q = R \oplus P$. Put

$$s = |E(P) \cap E(R)|.$$

Since Q has six edges,

$$6 = |E(R) \triangle E(P)| = 7 + 5 - 2s,$$

so $s = 3$.

For each $v \in V(P)$, let d_v be the number of common edges in $E(P) \cap E(R)$ incident with v. Since both P and R have degree two at v,

$$\deg_{R \oplus P}(v) = 4 - 2d_v.$$

Every vertex has degree zero or two in the cycle Q, hence $d_v \in \{1, 2\}$. Moreover,

$$\sum_{v \in V(P)} d_v = 2s = 6.$$

Exactly one of the five values d_v is therefore 2, and the other four are 1. Consequently,

$$P \cap R \cong P_3 \,\dot{\cup}\, K_2 :$$

the three common rim edges form a two-edge path and a disjoint edge.

By dihedral symmetry, take the two-edge path to be $6 - 0 - 1$. The disjoint rim edge is one of 23, 34, and 45. Reflection $x \mapsto -x$ exchanges the cases 23 and 45.

If the disjoint edge is 23, the two path components must be joined by two non-rim edges between $\{6, 1\}$ and $\{2, 3\}$. Among the four cross-pairs, 62 and 63 are nonedges of G, while 12 is a rim edge and would make $s = 4$. Only 13 remains, so no such 5-cycle P exists. The case 45 is symmetric.

If the disjoint edge is 34, the only possible joining edges are 13 and 46. Thus

$$P = 60134.$$

Direct cancellation gives

$$R \oplus P = 123 \,\dot{\cup}\, 456,$$

the disjoint union of two triangles, rather than a 6-cycle. This is again a contradiction.

Thus R has no incremental decomposition. Since G is pentangulated, this completes the proof of Theorem 1.1. $\square$

References

[1] T. A. McKee, *Pentangulated graphs*, AKCE Int. J. Graphs Comb. **7** (2010), 113–119. DOI: 10.1080/09728600.2010.12088917.
[2] D. B. West, *Pentagulated graphs*, REGS open problems page (2009), available online.