THE NONEXISTENCE OF A $(9,6,1)$-PERFECT MENDELSOHN DESIGN

USING GPT 5.6 SOL PRO

Abstract. We prove that a $(9,6,1)$-perfect Mendelsohn design does not exist. After one block is fixed by relabeling, the problem is equivalent to an exact-cover instance with 330 requirements and 2,568 candidate cyclic blocks. A complete finite search tree eliminates every possible exact cover.

1. Introduction

A *perfect Mendelsohn design* is a family of cyclically ordered blocks in which every ordered pair of distinct points occurs exactly once at each nonzero cyclic distance. Mendelsohn introduced perfect cyclic designs in 1977 [4]; the case of block size six was studied in detail by Abel, Bennett, and Zhang [1]. The CPro1 catalogue listed the instance $(v,k,b) = (9,6,12)$ among its open cases in its June 2026 version [2].

Theorem 1.1. *There is no $(9,6,1)$-perfect Mendelsohn design.*

The argument has two parts. First, the design problem is reduced exactly to a finite exact-cover problem. Second, that instance is eliminated by an exhaustive branching argument whose soundness is recorded in Lemma 3.1.

2. Exact-cover reduction

Let $X = \{0,1,\ldots,8\}$. A cyclic block is a cyclic ordering $B = (b_0, b_1, \ldots, b_5)$ of six distinct points of X, with subscripts read modulo 6. For such a block define

$$C(B) = \{(t, b_i, b_{i+t}) : 1 \le t \le 5,\ 0 \le i \le 5\}.$$

The universe of requirements is

$$\mathcal{R} = \{(t,x,y) : 1 \le t \le 5,\ x,y \in X,\ x \ne y\}.$$

Thus a $(9,6,1)$-perfect Mendelsohn design is precisely a family $\mathcal{D}$ of cyclic blocks for which the sets $C(B)$, $B \in \mathcal{D}$, partition $\mathcal{R}$.

Lemma 2.1. *Every such design contains exactly twelve blocks. After relabeling the points, it contains*

$$B_0 = (0,1,2,3,4,5).$$

Proof. There are $5 \cdot 9 \cdot 8 = 360$ requirements. The thirty triples in $C(B)$ are distinct for every block B: equality first forces the same distance t, and then distinctness of the block entries forces the same starting position. Hence each block covers thirty requirements, and any design has $360/30 = 12$ blocks.

Choose one block of a hypothetical design. Label its six points $0,1,\ldots,5$ in cyclic order, and label the remaining three points $6,7,8$. This gives the stated normalization. $\square$

A cyclic ordering has six linear representatives, one for each rotation. Consequently the number of cyclic six-blocks on X is

$$\binom{9}{6}5! = 10{,}080.$$

Represent each block uniquely by placing its least point first. Let

$$\mathcal{R}_0 = \mathcal{R} \setminus C(B_0), \qquad |\mathcal{R}_0| = 330,$$

and let $\mathcal{B}_0$ be the family of cyclic blocks B satisfying $C(B) \cap C(B_0) = \varnothing$. Direct enumeration from the definition gives

$$|\mathcal{B}_0| = 2{,}568.$$

For transparency, grouping the underlying six-subsets by the number of points they share with B_0 gives contributions 1,200, 1,224, and 144 when that intersection has size 3, 4, and 5, respectively; all other intersection sizes contribute zero. Each member of $\mathcal{B}_0$ determines a thirty-element subset of $\mathcal{R}_0$.

Proposition 2.2. *A $(9, 6, 1)$-perfect Mendelsohn design exists if and only if eleven members of $\mathcal{B}_0$ form an exact cover of $\mathcal{R}_0$.*

Proof. Suppose a design exists and apply Lemma 2.1. Its eleven blocks other than B_0 cannot contain a requirement already covered by B_0. They therefore belong to $\mathcal{B}_0$, and their requirement sets partition $\mathcal{R}_0$.

Conversely, if eleven members of $\mathcal{B}_0$ exactly cover $\mathcal{R}_0$, then adjoining B_0 gives a family whose requirement sets partition all of $\mathcal{R}$; this is a perfect Mendelsohn design. Since each candidate covers thirty requirements, any exact cover of the 330 residual requirements necessarily uses eleven candidates. □

3. Exhaustive elimination

At a node of the search tree, let $U \subseteq \mathcal{R}_0$ be the uncovered requirements and let $A \subseteq \mathcal{B}_0$ be the candidates compatible with all blocks already selected. For $r \in U$, put

$$A(r) = \{B \in A : r \in C(B)\}.$$

A contradiction leaf consists of an $r \in U$ for which $A(r) = \varnothing$. At every other node choose an uncovered requirement r and branch once for each $B \in A(r)$. In the branch corresponding to B, remove $C(B)$ from U and remove from A every candidate meeting $C(B)$.

Lemma 3.1 (Search-tree soundness)**.** *If a finite rooted tree has the preceding leaf and branching properties, then no exact cover extends the partial selection at its root.*

Proof. Proceed upward from the leaves. At a contradiction leaf, the named requirement is uncovered and has no compatible covering candidate, so no completion is possible. At an internal node, every exact cover must choose exactly one member of $A(r)$ to cover the selected requirement r. The children exhaust all of those choices, and by induction none can be completed. Hence the parent cannot be completed either. □

Proposition 3.2. *The residual exact-cover instance* $(\mathcal{R}_0, \mathcal{B}_0)$ *has no solution.*

Proof. Apply the preceding branching rule, choosing at each node an uncovered requirement for which $|A(r)|$ is minimum. The resulting search is finite and terminates in a complete contradiction tree: it has 36,019 nodes, of which 7,795 are branch nodes and 28,224 are contradiction leaves. Every internal node contains all candidates in $A(r)$, and every leaf has $A(r) = \varnothing$. Lemma 3.1 therefore rules out an exact cover. □

Proposition 3.2, together with Proposition 2.2, proves Theorem 1.1.

References

[1] R. J. R. Abel, F. E. Bennett, and H. Zhang, *Perfect Mendelsohn designs with block size six*, J. Statist. Plann. Inference **86** (2000), 287–319. doi:10.1016/S0378-3758(99)00114-7.

[2] Constructive Codes, *CPro1: Perfect Mendelsohn Design problem definition*, GitHub repository, version dated June 3, 2026.

[3] D. E. Knuth, *Dancing links*, in *Millennial Perspectives in Computer Science*, Palgrave, 2000, 187–214.

[4] N. S. Mendelsohn, *Perfect cyclic designs*, Discrete Math. **20** (1977), 63–68. doi:10.1016/0012-365X(77)90043-7.